

Strong solutions of semilinear parabolic equations with measure data and generalized backward stochastic differential equations*

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Abstract: We prove that under natural assumptions on the data strong solutions in Sobolev spaces of semilinear parabolic equations in divergence form involving measure on the right-hand side may be represented by solutions of some generalized backward stochastic differential equations. As an application we provide stochastic representation of strong solutions of the obstacle problem by means of solutions of some reflected backward stochastic differential equations. To prove the latter result we use a stochastic homographic approximation for solutions of the reflected backward equation. The approximation may be viewed as a stochastic analogue of the homographic approximation for solutions to the obstacle problem.

Keywords: Semilinear parabolic equation, Measure data, Obstacle problem, Strong solution, Generalized BSDE, Reflected BSDE, Homographic approximation.

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1 Introduction

Let μ be a Radon measure on $Q_T \equiv [0, T] \times \mathbb{R}^d$ and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, $f : Q_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions. In the paper we consider strong solutions in Sobolev spaces of the Cauchy problem

$$\frac{\partial u}{\partial t} + L_t u = -f_u - g(u)\mu, \quad u(T) = \varphi. \quad (1.1)$$

Here

$$L_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a^{ij} \frac{\partial}{\partial x_j}) + \sum_{i=1}^d b^i \frac{\partial}{\partial x_i} \quad (1.2)$$

is an operator with measurable coefficients $a : Q_T \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, $b : Q_T \rightarrow \mathbb{R}^d$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a^{ij}(t, x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad a^{ij} = a^{ji}, \quad |b^i(t, x)| \leq \Lambda, \quad \xi \in \mathbb{R}^d \quad (1.3)$$

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for some $0 < \lambda \leq \Lambda$, and $f_u(t, x) = f(t, x, u(t, x), \sigma \nabla u(t, x))$ with σ such that $\sigma \sigma^* = a$, $g(u)(t, x) = g(t, x, u(t, x))$, $(t, x) \in Q_T$.

Nonlinear elliptic equations in divergence form with measure data on the right-hand side are considered in [17]. Our interest in general parabolic equations of the form (1.1) with nonlinear g comes from that fact that as we shall see in Section 5 they arise naturally when considering the obstacle problem (parabolic variational inequalities). Let us mention also that equations of the form (1.1) include the so-called Schrödinger equations with measure data, that is parabolic equations of the form (1.1) with $g(t, x, y) = y$ considered for example in [10]. Abstract parabolic evolution equations involving measures which depend nonlinearly on the solution are considered in [1].

Let $\mathbb{X} = \{(X, P_{s,x}); (s, x) \in [0, T) \times \mathbb{R}^d\}$ be a Markov family corresponding to the operator L_t (see [24, 27]). Our main result concerning (1.1) says that if μ belongs to the weighted Sobolev space $\mathbb{L}_2(0, T; H_\rho^{-1})$ then under natural conditions on φ, f, g there exists a minimal strong solution of (1.1) and the pair

$$(Y_t^{s,x}, Z_t^{s,x}) = (u(t, X_t), \sigma \nabla u(t, X_t)), \quad t \in [s, T] \quad (1.4)$$

is a minimal solution of the generalized backward stochastic differential equation (GB-SDE)

$$\begin{aligned} Y_t^{s,x} &= \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x}) d\theta + \int_t^T g(\theta, X_\theta, Y_\theta^{s,x}) dR_{s,\theta} \\ &\quad - \int_t^T Z_\theta^{s,x} dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \end{aligned} \quad (1.5)$$

where $B_{s,\cdot}$ is some standard Wiener process under $P_{s,x}$ and R is a continuous additive functional of \mathbb{X} corresponding to μ in the sense that

$$E_{s,x} \int_s^T \eta(t, X_t) dR_t = \int_s^T \int_{\mathbb{R}^d} \eta(t, y) p(s, x, t, y) d\mu(t, y) \quad (1.6)$$

for any bounded measurable $\eta : Q_T \rightarrow [0, \infty)$. Here $E_{s,x}$ denotes expectation with respect to $P_{s,x}$ and p is the transition density function of $(X, P_{s,x})$ (or, equivalently, p is the fundamental solution for L_t). From (1.4) it follows in particular that $u(s, x) = Y_s^{s,x}$, so (1.5) may be viewed as the Feynman-Kac formula for solutions of (1.1).

In [7] it is proved that viscosity solutions of the Cauchy problem for semilinear parabolic equation in nondivergence form with obstacle can be represented by solutions of some reflected backward stochastic differential equations (RBSDEs). As an application of results concerning (1.1) we provide such a representation in the case where the equation is in divergent form and strong solutions in Sobolev spaces are considered. We strengthen also known analytical results on homographic approximation of solution of the obstacle problem.

Roughly speaking, the obstacle problem consists in finding $u : Q_T \rightarrow \mathbb{R}$ such that for given φ, f as above and $h : Q_T \rightarrow \mathbb{R}$,

$$\begin{cases} \min(u - h, -\frac{\partial u}{\partial t} - L_t u - f_u) = 0 & \text{in } Q_T, \\ u(T) = \varphi & \text{on } \mathbb{R}^d, \end{cases} \quad (1.7)$$

i.e. u satisfies the prescribed terminal condition, takes values above a given obstacle h , satisfies inequality $\frac{\partial u}{\partial t} + L_t u \leq -f_u$ in Q_T and equation $\frac{\partial u}{\partial t} + L_t u = -f_u$ on the set $\{u > h\}$.

In the case where L_t is a non-divergent operator of the form

$$L_t = \frac{1}{2} \sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i \frac{\partial}{\partial x_i},$$

problem (1.7) has been investigated carefully in [7] by using probabilistic methods. Let $X^{s,x}$ be a solution of the Itô equation

$$dX_t^{s,x} = \sigma(t, X_t^{s,x}) dW_t + b(t, X_t^{s,x}) dt, \quad X_s^{s,x} = x \quad (\sigma \sigma^* = a)$$

associated with L_t . In [7] it is proved, that under suitable assumptions on a, b and the data φ, f, h , for each $(s, x) \in Q_T$ there exists a unique solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of RBSDE with forward driving process $X^{s,x}$, terminal condition $\varphi(X_T^{s,x})$, coefficient f and obstacle $h(\cdot, X^{s,x})$, and moreover, u defined by the formula $u(s, x) = Y_s^{s,x}$, $(s, x) \in Q_T$ is a unique viscosity solution of (1.7) in the class of functions satisfying the polynomial growth condition.

In the present paper we are interested in stochastic representation of solutions of the obstacle problem with divergence form operator in the framework of Sobolev spaces (for the case of non-divergence form operator see [3], [18]). The advantage of using such a framework lies in the fact that it allows to provide stochastic representation not only for $Y^{s,x}$ but also for $Z^{s,x}$ and $K^{s,x}$.

By the strong solution of the obstacle problem we understand a pair (u, μ) consisting of a measurable function $u : Q_T \rightarrow \mathbb{R}$ having some regularity properties and a Radon measure μ on Q_T such that

$$\frac{\partial u}{\partial t} + L_t u = -f_u - \mu, \quad u(T) = \varphi, \quad u \geq h, \quad \int_{Q_T} (u - h) d\mu = 0 \quad (1.8)$$

(see Section 4 for details).

Let $S_t = h(t, X_t)$, $t \in [s, T]$, and let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ be a solution of RBSDE

$$\begin{cases} Y_t^{s,x} = \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x}) d\theta + K_T^{s,x} - K_t^{s,x} \\ \quad - \int_t^T Z_\theta^{s,x} dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \\ Y_t^{s,x} \geq S_t, \quad t \in [s, T], \\ K^{s,x} \text{ increasing, continuous, } K_s^{s,x} = 0, \quad \int_s^T (Y_t^{s,x} - S_t) dK_t^{s,x} = 0. \end{cases} \quad (1.9)$$

We show that under mild conditions on φ, f and h there exists a unique solution (u, μ) of (1.8). Moreover, for a.e. $(s, x) \in [0, T) \times \mathbb{R}^d$,

$$u(t, X_t) = Y_t^{s,x}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad Z_t^{s,x} = \sigma \nabla u(t, X_t), \quad \lambda \otimes P_{s,x}\text{-a.s.} \quad (1.10)$$

and $K^{s,x}$ corresponds to μ , i.e. (1.6) with R replaced by $K^{s,x}$ holds true. The correspondence between $K^{s,x}$ and μ allows us to derive properties of $K^{s,x}$ from those of μ and vice versa.

Our proof of (1.10) and the correspondence between $K^{s,x}$ and μ is based on a general approximation result for solutions of RBSDEs. The approximation we consider may be viewed as an analogue of the well known in PDEs theory homographic approximation for strong solutions of an obstacle problem (see [20]). Therefore we call it a stochastic homographic approximation. Up to our knowledge, it is used here for the first time in the context of RBSDEs.

By using the stochastic homographic approximation we prove also that under mild regularity conditions on h the measure μ is absolutely continuous with respect to the Lebesgue measure λ on \mathbb{R}^d , and we get some information on the density $d\mu/d\lambda$. This provides information on the density of the control process $K^{s,x}$. It is worth pointing out that the approximation provides additional information on the control process also for general non-Markovian RBSDE with obstacle being a general continuous semimartingale. For instance, it allows to prove a stochastic version of the Lewy-Stampacchia inequality.

Our results on convergence of stochastic homographic approximations to solutions of (1.9) when combined with (1.10) prove convergence of homographic approximations of solutions of (1.8). In particular, we show that if $\frac{\partial h}{\partial t} + L_t h$ is a signed Radon measure on Q_T then under some assumptions on φ, f , the strong solution u of (1.8) is a limit, in the space $\mathbb{L}_2(0, T; H_\rho^1) \cap C([0, T], \mathbb{L}_2(\mathbb{R}^d))$, of maximal solutions of the problem

$$\frac{\partial u_n}{\partial t} + L_t u_n = -f_{u_n} - \mu_n, \quad u_n(T) = \varphi$$

with

$$\mu_n = \frac{1}{1 + n|u_n - h|} \Phi^-, \quad \Phi = \frac{\partial h}{\partial t} + L_t h + f_h.$$

This strengthens considerably analytical results which asserts that u is approximated by homographic approximations in $\mathbb{L}_{2,\rho}(Q_T)$, while its gradient weakly in $\mathbb{L}_{2,\rho}(Q_T)$ (see [20]). Let us point out also that contrary to [20] our approximation is direct in the sense that it does not require smoothing the functional Φ^- . Furthermore, we prove that $\{\mu_n\}$ converges to μ weakly and in the dual space to the space \mathcal{W}_ρ (see notation below).

Finally, let us mention that in case $b = 0$ from our stochastic Lewy-Stampacchia inequality we get easily the Lewy-Stampacchia inequality for solutions of the obstacle problem (1.8).

In the paper we adopt the following notation.

$$Q_T = [0, T] \times \mathbb{R}^d, \quad Q_{\hat{T}} = [0, T] \times \mathbb{R}^d, \quad \check{Q}_T = (0, T) \times \mathbb{R}^d, \quad \nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}).$$

By $\mathcal{B}(D), \mathcal{B}_b(D), \mathcal{B}^+(D)$ we denote the set of Borel, bounded Borel, positive Borel functions on D respectively. $C_0(D), C_0^\infty(D), C_b^\infty(D)$ are spaces of all continuous functions with compact support in D , smooth functions with compact support in D and smooth functions on D with bounded derivatives, respectively. We also write that $K \subset\subset D$ if K is compact and included in D .

$\mathbb{L}_p(\mathbb{R}^d)$ ($\mathbb{L}_p(Q_T)$) are usual Banach spaces of measurable functions on \mathbb{R}^d (on Q_T) that are p -integrable. Let ρ be a positive function on \mathbb{R}^d . By $\mathbb{L}_{p,\rho}(\mathbb{R}^d)$ ($\mathbb{L}_{p,\rho}(Q_T)$) we denote the space of functions u such that $u\rho \in \mathbb{L}_p(\mathbb{R}^d)$ ($u\rho \in \mathbb{L}_p(Q_T)$) equipped with the norm $\|u\|_{p,\rho} = \|u\rho\|_p$ ($\|u\|_{p,\rho,T} = \|u\rho\|_{p,T}$). By $\langle \cdot, \cdot \rangle_{2,\rho}$ we denote the inner product in $\mathbb{L}_{2,\rho}(\mathbb{R}^d)$ and by $\langle \cdot, \cdot \rangle_{2,\rho,T}$ the inner product in $\mathbb{L}_{2,\rho}(Q_T)$.

H_ϱ^1 is the Banach space consisting of all elements u of $\mathbb{L}_{2,\varrho}(\mathbb{R}^d)$ having generalized derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \dots, d$, in $\mathbb{L}_{2,\varrho}(\mathbb{R}^d)$. \mathcal{W}_ϱ is the subspace of $\mathbb{L}_2(0, T; H_\varrho^1)$ consisting of all elements u such that $\frac{\partial u}{\partial t} \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, where H_ϱ^{-1} is the dual space to H_ϱ^1 (see [15] for details). By $\langle \cdot, \cdot \rangle_{\varrho, T}$ we denote duality between $\mathbb{L}_2(0, T; H_\varrho^{-1})$ and $\mathbb{L}_2(0, T; H_\varrho^1)$. $\mathcal{M}(Q_T)$ ($\mathcal{M}^+(Q_T)$) denotes the space of Radon measures (positive Radon measures) on Q_T . By m we denote the Lebesgue measure on Q_T .

By C we denote a general constant which may vary from line to line but depends only on fixed parameters.

2 Generalized BSDEs

Let $\{B_t, 0 \leq t \leq T\}$ be a d -dimensional standard Wiener process defined on some probability space (Ω, \mathcal{F}, P) and let $\{\mathcal{F}_t\}_{t \in [0, T]}$ denote the usual augmentation of the natural filtration generated by B .

Let ξ be an \mathcal{F}_T -measurable random variable and let $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$. We will need the following assumptions.

(A1) $\xi \in \mathbb{L}_2(\Omega, \mathcal{F}_T, P)$,

(A2) For every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ the processes $f(\cdot, y, z)$, $g(\cdot, y)$ are predictable,

(A3) R is increasing continuous process such that $E|R_T|^2 < \infty$,

(A4) There exist $K > 0$ and a predictable process γ such that $E \int_0^T |\gamma_t|^2 dt < \infty$ and

$$|f(t, y, z)| \leq K(|\gamma_t| + |y| + |z|), \quad P\text{-a.s.}$$

for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$

(A5) There exists $M > 0$ such that $|g(t, \cdot)| \leq M$, $t \in [0, T]$, P -a.s.,

(A6) $(y, z) \rightarrow f(t, y, z)$ is P -a.s. continuous for every $t \in [0, T]$,

(A7) $y \rightarrow g(t, y)$ is P -a.s. continuous for every $t \in [0, T]$,

(A6') There is $L > 0$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|), \quad P\text{-a.s.}$$

for every $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $t \in [0, T]$,

(A7') There is $L > 0$ such that

$$|g(t, y_1) - g(t, y_2)| \leq L|y_1 - y_2|, \quad P\text{-a.s.}$$

for every $y_1, y_2 \in \mathbb{R}$, $t \in [0, T]$.

Following [21, 22] we say that a pair (Y, Z) of $\{\mathcal{F}_t\}$ -progressively measurable processes on $[0, T]$ taking values in $\mathbb{R} \times \mathbb{R}^d$ is a solution of the generalized backward stochastic differential equation (GBSDE)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(t, Y_t) dR_t - \int_t^T (Z_s, dB_s), \quad t \in [0, T] \quad (2.1)$$

if $E \sup_{0 \leq t \leq T} |Y_t|^2 < \infty$, $E \int_0^T |Z_t|^2 dt < \infty$ and (2.1) is satisfied P -a.s. If (Y, Z) is a solution of (2.1) such that $Y_t \leq \tilde{Y}_t$, $t \in [0, T]$, P -a.s. for any solution (\tilde{Y}, \tilde{Z}) of (2.1), then it is called a minimal solution of (2.1).

Of course, if $R = 0$ or $g = 0$, then GBSDE reduces to the usual backward SDE with terminal condition ξ and coefficient f .

The main purpose of the present section is to prove comparison results for solutions of (2.1), and, in consequence, to prove that under (A1)–(A7) there is a minimal solution to (2.1).

We begin with a priori estimates for solutions of (2.1) and a "backward version" of the generalized Gronwall's lemma, which in turn will be used to prove some comparison principle for solutions of GBSDEs. Let us mention here that a priori estimates and comparison results for solutions of GBSDEs are proved in [21] but under assumptions on g not suitable for our purposes (in [21] monotonicity of g is required).

Proposition 2.1. *Assume (A1)–(A5) and let (Y, Z) be a solution of (2.1). Then there exists $C > 0$ depending on K, M, T such that*

$$E \sup_{0 \leq t \leq T} |Y_t|^2 + E \int_0^T |Z_t|^2 dt \leq C(E|\xi|^2 + E|R_T|^2 + E \int_0^T |\gamma_t|^2 dt).$$

Proof. By Itô's formula, for every $t \in [0, T]$,

$$|Y_t|^2 + \int_t^T |Z_\theta|^2 d\theta = |\xi|^2 + \int_t^T f(\theta, Y_\theta, Z_\theta) Y_\theta d\theta + \int_t^T g(Y_\theta) Y_\theta dR_\theta - \int_t^T Z_\theta Y_\theta dB_\theta.$$

Hence, by (A4) and (A5),

$$\begin{aligned} |Y_t|^2 + \frac{1}{2} \int_t^T |Z_\theta|^2 d\theta &\leq |\xi|^2 + 2(K + K^2) \int_t^T |Y_\theta|^2 d\theta \\ &\quad + K \int_t^T |\gamma_\theta|^2 d\theta + M \int_t^T |Y_\theta| dR_\theta - \int_t^T Z_\theta Y_\theta dB_\theta. \end{aligned} \quad (2.2)$$

Taking expectation and using Gronwall's lemma yields

$$E|Y_t|^2 + E \int_0^T |Z_\theta|^2 d\theta \leq C(E|\xi|^2 + E \int_0^T |\gamma_t|^2 dt + E \int_0^T |Y_\theta| dR_\theta). \quad (2.3)$$

Therefore taking supremum in (2.2), using the Burkholder-Davis-Gundy inequality and (2.3) we get

$$\begin{aligned} E \sup_{0 \leq t \leq T} |Y_t|^2 + E \int_0^T |Z_\theta|^2 d\theta &\leq C(E\xi^2 + E \int_0^T |\gamma_t|^2 dt + E \int_0^T |Y_\theta| dR_\theta) \\ &\leq C(E|\xi|^2 + E \int_0^T |\gamma_t|^2 dt + E|R_T|^2) + \frac{1}{2} E \sup_{0 \leq t \leq T} |Y_t|^2, \end{aligned}$$

which proves the proposition. \square

Lemma 2.2. *Let Y be a continuous decreasing process such that $Y \geq 0$ a.s. and $EY_0 < \infty$, and let A be an adapted continuous increasing processes such that $0 \leq A_T \leq a$ a.s. for some $a > 0$. If*

$$EY_\tau \leq E \int_\tau^T Y_s dA_s$$

for every stopping time $0 \leq \tau \leq T$ then $Y = 0$.

Proof. Without lost of generality we may and will assume that A is strictly increasing. Put $\tau_t = \inf\{s \in [0, T]; A_s \geq t\} \wedge T$. By the change of variable formula and assumptions,

$$\begin{aligned} EY_{\tau_t} &\leq E \int_{\tau_t}^T Y_s dA_s = E \int_0^T Y_s \mathbf{1}_{\{\tau_t \leq s \leq T\}} dA_s \\ &\leq E \int_0^\infty Y_{\tau_u} \mathbf{1}_{\{\tau_t \leq \tau_u \leq T\}} du = \int_t^{A_T} EY_{\tau_u} du \leq \int_t^a EY_{\tau_u} du. \end{aligned}$$

and the result follows by classical Gronwall's lemma. \square

Theorem 2.3. *Suppose that ξ_i, f_i, g_i, R^i , $i = 1, 2$ satisfy (A1)-(A5) and, in addition, f_1, g_1 satisfy (A6'), (A7'). Let (Y^i, Z^i) be a solution of (2.1) with data ξ_i, f_i, g_i, R^i , $i = 1, 2$. If*

- (i) $\xi_1 \leq \xi_2$, P -a.s.,
- (ii) $f_1(\cdot, y, z) \leq f_2(\cdot, y, z)$, $dt \otimes dP$ -a.e. for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,
- (iii) $g_1(t, Y_t^2) dR_t^1 \leq g_2(t, Y_t^2) dR_t^2$, P -a.s.,
- (iv) $R_T^1 \leq a$, P -a.s. for some $a > 0$,

then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$, P -a.s..

Proof. Write $\xi = \xi_1 - \xi_2$, $Y = Y^1 - Y^2$, $Z = Z^1 - Z^2$. By Itô's formula and assumptions,

$$\begin{aligned} |Y_t^+|^2 &+ \int_t^T |Z_\theta|^2 \mathbf{1}_{\{Y_\theta > 0\}} d\theta \\ &= |\xi^+|^2 + 2 \int_t^T (f_1(\theta, Y_\theta^1, Z_\theta^1) - f_2(\theta, Y_\theta^2, Z_\theta^2)) Y_\theta^+ d\theta \\ &\quad + \int_t^T g_1(\theta, Y_\theta^1) Y_\theta^+ dR_\theta^1 - \int_t^T g_2(\theta, Y_\theta^2) Y_\theta^+ dR_\theta^2 - \int_t^T Z_\theta Y_\theta^+ dB_\theta \\ &\leq C \int_t^T |Y_\theta^+|^2 + \alpha \int_t^T |Z_\theta|^2 \mathbf{1}_{\{Y_\theta > 0\}} d\theta + \int_t^T (g_1(\theta, Y_\theta^1) - g_1(\theta, Y_\theta^2)) Y_\theta^+ dR_\theta^1 \\ &\quad - \int_t^T Z_\theta Y_\theta^+ dB_\theta. \end{aligned}$$

Now, arguing as in the proof of Proposition 2.1 we get

$$E \sup_{t \leq s \leq T} |Y_s^+|^2 \leq C \left(E \int_t^T |Y_\theta^+|^2 d\theta + E \int_t^T |g_1(\theta, Y_\theta^1) - g_1(\theta, Y_\theta^2)| Y_\theta^+ dR_\theta^1 \right).$$

Hence,

$$E \sup_{t \leq s \leq T} |Y_s^+|^2 \leq CE \int_t^T \sup_{s \leq \theta \leq T} |Y_\theta^+|^2 (ds + dR_s^1).$$

The same inequality we can get for every stopping time $\tau \leq T$ instead of t . By Lemma 2.2 we get the result. \square

Let \mathbb{Q} denote the set of rational numbers. The following useful approximation result is proved in [14].

Lemma 2.4. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function such that $|f(x)| \leq C(1 + |x|)$, $x \in \mathbb{R}^d$, for some $C > 0$. Set $f_n(x) = \inf_{y \in \mathbb{Q}^d} \{f(y) + n|x - y|\}$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. Then*

- (a) $|f_n(x)| \leq C(1 + |x|)$, $x \in \mathbb{R}^d$,
- (b) $f_n(x) \uparrow f(x)$, $x \in \mathbb{R}^d$,
- (c) if $x_n \rightarrow x$ then $f_n(x_n) \rightarrow f(x)$,
- (d) f_n is Lipschitz continuous.

We will need also the following lemma.

Lemma 2.5. *Let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a continuous function. Then there is a sequence $\{f_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that*

- (a) $f_n(x) \uparrow f(x)$, $x \in \mathbb{R}^d$,
- (b) if $x_n \rightarrow x$ then $f_n(x_n) \rightarrow f(x)$.

Proof. By the Stone-Weierstrass theorem, for every $\varepsilon > 0$ there is $\tilde{f}_\varepsilon \in C^\infty(Q_T)$ such that $\|\tilde{f}_\varepsilon - f\|_\infty \leq \varepsilon$. Let $\bar{f}_n = \tilde{f}_{4^{-n}} - 2 \cdot 4^{-n}$. Then $\bar{f}_n \leq \bar{f}_{n+1}$, $n \geq 1$, because $\bar{f}_n \leq f$ and $4^{-n} \leq f - \bar{f}_n \leq 3 \cdot 4^{-n}$ for $n \geq 1$. Therefore the sequence $\{f_n = \eta_n \bar{f}_n\}$, where $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^d)$ is a sequence of positive functions such that $\eta_n \uparrow 1$ uniformly in compacts subsets of \mathbb{R}^d has the desired properties. \square

Proposition 2.6. *If assumptions (A1)–(A7) are satisfied then there exists a minimal solution of GBSDE (2.1). Moreover, if ξ_i, f_i, g_i, R^i , $i = 1, 2$, satisfy assumptions (i)–(iii) of Theorem 2.3 and the pairs (Y^i, Z^i) , $i = 1, 2$, are minimal solutions of (2.1) with data ξ_i, f_i, g_i, R^i , respectively, then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$, P -a.s..*

Proof. Let f_n be the approximation of f considered in Lemma 2.4 and let g_n be the approximation of g considered in Lemma 2.5. From [21] we know that for each $n \in \mathbb{N}$ there exists a unique solution (Y^n, Z^n) of GBSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + \int_t^T g_n(t, Y_t^n) dR_t^n - \int_t^T (Z_s^n, dB_s), \quad t \in [0, T] \quad (2.4)$$

with $R^n = R \wedge n$. By Theorem 2.3, $\{Y^n\}$ is increasing, and by Proposition 2.1,

$$E \sup_{0 \leq t \leq T} |Y_t^n|^2 + E \int_0^T |Z_t^n|^2 dt \leq C \left(E|\xi|^2 + E|R_T|^2 + E \int_0^T |\gamma_t|^2 dt \right) \quad (2.5)$$

for some C not depending on n . Therefore,

$$E \int_0^T |Y_t^n - Y_t^m|^2 dt + E \int_0^T |Y_t^n - Y_t^m| dR_t \rightarrow 0.$$

By Itô's formula,

$$\begin{aligned} & |Y_t^n - Y_t^m|^2 + \int_t^T |Z_\theta^n - Z_\theta^m|^2 d\theta \\ &= \int_t^T (Y_\theta^n - Y_\theta^m)(f_n(\theta, Y_\theta^n, Z_\theta^n) - f_n(\theta, Y_\theta^m, Z_\theta^m)) d\theta \\ &+ \int_t^T (Y_\theta^n - Y_\theta^m)g_n(\theta, Y_\theta^n) dR_\theta^n - \int_t^T (Y_\theta^n - Y_\theta^m)g_m(\theta, Y_\theta^m) dR_\theta^m \\ &+ \int_t^T (Y_\theta^n - Y_\theta^m)(Z_\theta^n - Z_\theta^m) dB_\theta. \end{aligned} \quad (2.6)$$

From the above and (2.5) we conclude that

$$\begin{aligned} & E|Y_t^n - Y_t^m|^2 + E \int_0^T |Z_t^n - Z_t^m|^2 dt \\ & \leq C \left(\left(E \int_0^T |Y_t^n - Y_t^m|^2 d\theta \right)^{1/2} + E \int_0^T |Y_t^n - Y_t^m| dR_t \right) \equiv I^{n,m}. \end{aligned} \quad (2.7)$$

Now taking supremum in (2.6), using BDG inequality and estimate (2.7) we get

$$E \sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 + E \int_0^T |Z_t^n - Z_t^m|^2 dt \leq I^{n,m}.$$

Since we know that $I^{n,m} \rightarrow 0$ as $n, m \rightarrow \infty$, passing to the limit in (2.4) proves existence of a solution (Y, Z) of (2.1). Furthermore, by Theorem 2.3, if (\tilde{Y}, \tilde{Z}) is a solution of (2.1) then $Y_t^n \leq \tilde{Y}_t$, $t \in [0, T]$, P -a.s. for each $n \in \mathbb{N}$, which implies that $Y \leq \tilde{Y}$.

To prove the second part of the theorem, we approximate (Y^1, Z^1) in the same manner as above. Let $\{(Y^{1,n}, Z^{1,n})\}$ denote the approximating sequence. By Theorem 2.3, $Y_t^{1,n} \leq Y_t^2$, $t \in [0, T]$, P -a.s. for $n \in \mathbb{N}$, which yields $Y^1 \leq Y^2$. \square

Remark 2.7. Under assumptions (A1)–(A7) there exists a maximal solution of GBSDE (2.1). This follows from the fact that if $\bar{f}(t, y, z) = -f(t, -y, -z)$, $\bar{g}(t, y) = -g(t, -y)$, and if (\bar{Y}, \bar{Z}) is a solution of (2.1) with ξ, f, g replaced by $-\xi, \bar{f}, \bar{g}$, then the pair $(-\bar{Y}, -\bar{Z})$ is a solution of (2.1). Therefore, if (Y, Z) is a minimal solution of (2.1) with data $-\xi, \bar{f}, \bar{g}$, then $(-Y, -Z)$ is a maximal solution of (2.1).

3 Stochastic homographic approximation

In what follows S denote a continuous $\{\mathcal{F}_t\}$ -progressively measurable real-valued process S on $[0, T]$ such that

$$(A8) \quad S_T \leq \xi \text{ } P\text{-a.s. and } E \sup_{0 \leq t \leq T} |S_t^+|^2 < \infty.$$

Recall that a triple (Y, Z, K) of $\{\mathcal{F}_t\}$ -progressively measurable processes on $[0, T]$ taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$ is a solution of the reflected backward stochastic differential equation (RBSDE)

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T (Z_s, dB_s), & t \in [0, T], \\ Y_t \geq S_t, & t \in [0, T], \\ K \text{ is increasing, continuous, } K_0 = 0, \int_0^T (Y_t - S_t) dK_t = 0 \end{cases} \quad (3.1)$$

if $E \sup_{0 \leq t \leq T} |Y_t|^2 < \infty$, $E \int_0^T |Z_t|^2 dt < \infty$, $E|K_T|^2 < \infty$ and (3.1) is satisfied P -a.s.

In [7] it is proved that if (A1), (A2), (A4), (A7') and (A8) are satisfied then (3.1) has a unique solution.

In the following theorem we assume that S is a continuous semimartingale admitting the decomposition

$$S_t = S_T + \int_t^T f(s, S_s, \tilde{Z}_s) ds - (C_T - C_t) + (R_T - R_t) - \int_t^T \tilde{Z}_s dB_s, \quad (3.2)$$

where \tilde{Z} is an $\{\mathcal{F}_t\}$ -adapted square-integrable process and C, R are continuous $\{\mathcal{F}_t\}$ -adapted square-integrable increasing processes.

Remark 3.1. If S is a semimartingale with the decomposition

$$S_t = S_0 + M_t + U_t, \quad t \in [0, T],$$

where M is square-integrable martingale on $[0, T]$ and U is an adapted process of square-integrable variation on $[0, T]$, then it admits decomposition of the form (3.2). Indeed, by the representation theorem for martingales, there is a progressively measurable process \tilde{Z} such that $E \int_0^T |\tilde{Z}_s|^2 ds < \infty$ and

$$M_t = \int_0^t \tilde{Z}_s dB_s, \quad t \in [0, T].$$

Moreover, since U is a finite variation process, there exist increasing processes U^+, U^- such that $U_t = U_t^+ - U_t^-$, $t \in [0, T]$. Therefore putting

$$C_t = \int_0^t (f(s, S_s, \tilde{Z}_s))^+ ds + U_t^+, \quad R_t = \int_0^t (f(s, S_s, \tilde{Z}_s))^- ds + U_t^-, \quad t \in [0, T]$$

yields (3.2).

Theorem 3.2. Assume (A1), (A2), (A4), (A7') and (A8). Let S be of the form (3.2) and for $n \in \mathbb{N}$ let (Y^n, Z^n) be a maximal solution of the following GBSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s, \quad t \in [0, T], \quad (3.3)$$

where

$$K_t^n = \int_0^t \alpha_s^n dR_s, \quad \alpha_t^n = \frac{1}{1 + n|Y_t^n - S_t|}. \quad (3.4)$$

Then $Y_t^n \geq S_t$, $t \in [s, T]$ a.s. for each $n \in \mathbb{N}$, $Y_t^n \downarrow Y_t$ a.s. for every $t \in [0, T]$ and

$$E \sup_{t \in [0, T]} |Y_t^n - Y_t|^2 + E \int_0^T |Z_t^n - Z_t|^2 dt + E \sup_{t \in [0, T]} |K_t^n - K_t|^2 \rightarrow 0, \quad (3.5)$$

where (Y, Z, K) is a solution of (3.1).

Proof. Let $(\bar{Y}_t^n, \bar{Z}_t^n)$ be a solution of (3.3) with $\bar{\alpha}_t^n = 1 - \frac{n(\bar{Y}_t^n - S_t)}{1+n|\bar{Y}_t^n - S_t|}$ in place of α^n . By Itô's formula, for every $t \in [0, T]$ we have

$$\begin{aligned} |(\bar{Y}_t^n - S_t)^-|^2 &= |(\bar{Y}_T^n - S_T)^-|^2 - 2 \int_t^T \frac{n|(\bar{Y}_s^n - S_s)^-|^2}{1+n|\bar{Y}_s^n - S_s|} dR_s \\ &\quad - 2 \int_t^T (\bar{Y}_s^n - S_s)^- dC_s - 2 \int_t^T (\bar{Y}_s^n - S_s)^- (\bar{Z}_s^n - \tilde{Z}_s) dB_s \\ &\quad - \int_t^T \mathbf{1}_{\{\bar{Y}_s^n - S_s \leq 0\}} |\bar{Z}_s^n - \tilde{Z}_s|^2 ds \\ &\quad + 2 \int_t^T (f(s, S_s, \tilde{Z}_s) - f(s, \bar{Y}_s^n, \bar{Z}_s^n)) (\bar{Y}_s^n - S_s)^- ds. \end{aligned}$$

From this we obtain

$$E|(\bar{Y}_t^n - S_t)^-|^2 \leq CE \int_t^T |(\bar{Y}_s^n - S_s)^-|^2 ds,$$

which implies that $S \leq \bar{Y}^n$. From this we see that (\bar{Y}^n, \bar{Z}^n) is a solutions of (3.3). From maximality of Y^n we have that $S \leq \bar{Y}^n \leq Y^n$ and we get (i). Using Itô's formula, the Burkholder-Davis-Gundy inequality and standard estimates we get

$$E \sup_{0 \leq t \leq T} |Y_t^n|^2 + E \int_0^T |Z_t^n|^2 dt \leq CE \left(|\xi|^2 + \int_0^T |\gamma_t|^2 dt + \int_0^T \alpha_t^n |Y_t^n| dR_t \right).$$

It follows from the form of equation (3.3) and Proposition 2.6 that $Y_t^n \geq Y_t^{n+1}$, $t \in [0, T]$, P -a.s., $n \in \mathbb{N}$. Hence

$$\begin{aligned} E \sup_{0 \leq t \leq T} |Y_t^n|^2 + E \int_0^T |Z_t^n|^2 dt \\ \leq CE \left(|\xi|^2 + \int_0^T |\gamma_t|^2 dt + \int_0^T \alpha_t^n |Y_t^n| dR_t + \int_0^T \alpha_t^n |S_t| dR_t \right). \end{aligned} \quad (3.6)$$

Using once again Itô's formula we get

$$\begin{aligned} E|Y_t^n - Y_t^m|^2 + E \int_t^T |Z_s^n - Z_s^m|^2 ds \\ = -2E \int_t^T (Y_s^n - Y_s^m)(f(s, Y_s^m, Z_s^m) - f(s, Y_s^n, Z_s^n)) ds \\ - E \int_t^T (Y_s^n - Y_s^m)(\alpha_s^m - \alpha_s^n) dR_s \\ \leq \frac{1}{2}E \int_t^T |Z_s^n - Z_s^m|^2 ds + CE \int_t^T |Y_s^n - Y_s^m|^2 ds \\ + E \int_0^T |\alpha_s^m - \alpha_s^n| |Y_s^n - Y_s^m| dR_s. \end{aligned}$$

By the above and Gronwall's lemma,

$$E|Y_t^n - Y_t^m|^2 + \int_0^T |Z_s^n - Z_s^m|^2 ds \leq CE \int_0^T |\alpha_s^m - \alpha_s^n| |Y_s^n - Y_s^m| dR_s. \quad (3.7)$$

From the monotonicity of $\{Y^n\}$ there is a process \bar{Y} such that $Y_t^n \searrow \bar{Y}_t$, $t \in [0, T]$. From this and (3.6) we conclude that $E \int_0^T |Y_t^n - Y_t^m|^2 dR_t \rightarrow 0$. Hence, by (3.7), $E \int_0^T |Z_t^n - Z_t^m|^2 dt \rightarrow 0$ as $n, m \rightarrow \infty$. Using the Burkholder-Davis-Gundy inequality we conclude from the above that

$$E \sup_{t \in [0, T]} |Y_t^n - Y_t^m|^2 + E \int_0^T |Z_t^n - Z_t^m| dt \leq C \int_0^T |\alpha_s^n - \alpha_s^m| |Y_s^n - Y_s^m| dR_s \rightarrow 0$$

as $n, m \rightarrow \infty$, and hence, by (3.3), that $E \sup_t |K_t^n - K_t^m|^2 \rightarrow 0$ as $n, m \rightarrow \infty$. This implies that there is a triple $(\bar{Y}, \bar{Z}, \bar{K})$ such that \bar{Y} is continuous, \bar{K} continuous and increasing, satisfying

$$E \sup_{t \in [0, T]} (Y_t^n - \bar{Y}_t)^2 + E \int_0^T |Z_t^n - \bar{Z}_t|^2 dt + E \sup_{t \in [0, T]} |K_t^n - \bar{K}_t|^2 \rightarrow 0.$$

From this we obtain that

$$\int_0^T \mathbf{1}_{\{\bar{Y}_s - S_s > 0\}} (Y_s^n - S_s) dK_s^n \rightarrow \int_0^T \mathbf{1}_{\{\bar{Y}_s - S_s > 0\}} (\bar{Y}_s - S_s) d\bar{K}_s = \int_0^T (\bar{Y}_s - S_s) d\bar{K}_s.$$

P -a.s.. On the other hand,

$$\int_0^T \mathbf{1}_{\{\bar{Y}_s - S_s > 0\}} (Y_s^n - S_s) dK_s^n = \int_0^T \mathbf{1}_{\{\bar{Y}_s - S_s > 0\}} \frac{(Y_s^n - S_s)}{1 + n(Y_s^n - S_s)} dR_s \rightarrow 0.$$

Accordingly, $\int_0^T (\bar{Y}_s - S_s) d\bar{K}_s = 0$, P -a.s.. Therefore, by uniqueness of solutions of RBSDEs, $(\bar{Y}, \bar{Z}, \bar{K}) = (Y, Z, K)$ \square

The following corollary may be viewed as a stochastic version of the Lewy-Stampacchia inequality (see [5, 20] and Remark 5.5).

Corollary 3.3. *Under assumptions of Theorem 3.2,*

$$0 \leq dK_t \leq \mathbf{1}_{\{Y_t = S_t\}} dR_t. \quad (3.8)$$

Proof. Follows from (3.4), (3.5). \square

Remark 3.4. If S is an Itô process of the form

$$S_t = S_0 + \int_0^t \tilde{Z}_s dB_s + \int_0^t U_s ds, \quad t \in [0, T],$$

where U, \tilde{Z} are progressively measurable processes such that $E \int_0^T (|U_t|^2 + |\tilde{Z}_t|^2) dt < \infty$, then by [7, Remark 4.3],

$$K_t = \int_0^t \mathbf{1}_{\{Y_s = S_s\}} \alpha_s(f(s, S_s, \tilde{Z}_s) + U_s)^- ds \quad (3.9)$$

for some progressively measurable process α with values in $[0, 1]$. In fact, from every subsequence $\{n'\}$ we may choose a further subsequence $\{n''\}$ such that $\alpha^{n''} \rightarrow \alpha$ weakly in $\mathbb{L}_2((0, T) \times \Omega; dt \otimes dP)$, which provides some additional information on α . To see that α can be approximated by α_n , let us first observe that S may be written in the form

$$S_t = S_T + \int_t^T f(s, S_s, Z_s) ds - (\tilde{C}_T - \tilde{C}_t) + (\tilde{R}_T - \tilde{R}_t) - \int_t^T Z_s dB_s, \quad t \in [0, T],$$

where

$$\tilde{C}_t = \int_0^t (f(s, S_s, \tilde{Z}_s) + U_s)^+ ds, \quad \tilde{R}_t = \int_0^t (f(s, S_s, \tilde{Z}_s) + U_s)^- ds.$$

Therefore, by Theorem 3.2, if Y^n, Z^n, α^n are defined by (3.3), (3.4) and K^n is defined by the formula

$$K_t^n = \int_0^t \alpha_s^n (f(s, S_s, \tilde{Z}_s) + U_s)^- ds,$$

then (3.5) holds true. Since α^n are uniformly bounded, there is a subsequence $\{n'\}$ such that $\alpha^{n'} \rightarrow \bar{\alpha}$ weakly in $\mathbb{L}_2((0, T) \times \Omega; dt \otimes dP)$. Since $\alpha_s^n (f(s, S_s, \tilde{Z}_s) + U_s)^-$ are uniformly bounded in $\mathbb{L}_2((0, T) \times \Omega; dt \otimes dP)$ as well, there is $\{n''\} \subset \{n'\}$ such that

$$\alpha_t^{n''} (f(t, S_t, \tilde{Z}_t) + U_t)^- \rightarrow \mathbf{1}_{\{Y_t=S_t\}} \alpha_t (f(t, S_t, \tilde{Z}_t) + U_t)^-, \quad \alpha^{n''} \rightarrow \bar{\alpha}$$

weakly in $\mathbb{L}_2((0, T) \times \Omega; dt \otimes dP)$. From this we conclude that $\bar{\alpha} = \alpha$ on the set $\{\mathbf{1}_{\{Y_t=S_t\}} \alpha_s (f(s, S_s, \tilde{Z}_s) + U_s)^- > 0\}$.

Remark 3.5. Analysis of the proof of Theorem 3.2 shows that the assumption that S is continuous is superfluous. What we really need is continuity of the process R . This is related to the fact, that if C is càdlàg and R is continuous then S has only downward jumps going backward in time (see [11]).

4 Semilinear parabolic equations with measure data

Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, $f : Q_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions. In this section we are concerned with existence and stochastic representation of a solution of the problem (1.1)

Let $\Omega = C([0, T], \mathbb{R}^d)$ denote the space of continuous \mathbb{R}^d -valued functions on $[0, T]$ equipped with the topology of uniform convergence and let X be a canonical process on Ω . It is known that for an operator L_t defined by (1.2) with a and b satisfying (1.3) one can construct a weak fundamental solution $p(s, x, t, y)$ for L_t and then a Markov family $\mathbb{X} = \{(X, P_{s,x}); (s, x) \in [0, T] \times \mathbb{R}^d\}$ for which p is the transition density function, i.e.

$$P_{s,x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_{\Gamma} p(s, x, t, y) dy, \quad t \in (s, T]$$

for any Γ in a Borel σ -field \mathcal{B} of \mathbb{R}^d (see [24, 27]).

Set $\mathcal{F}_t^s = \sigma(X_u, u \in [s, t])$, $\bar{\mathcal{F}}_t^s = \sigma(X_u, u \in [T + s - t, T])$ and define \mathcal{G} as the completion of \mathcal{F}_T^s with respect to the family $\mathcal{P} = \{P_{s,\mu} : \mu \text{ is a probability measure on}$

$\mathcal{B}(\mathbb{R}^d)\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$, and define \mathcal{G}_t^s ($\bar{\mathcal{G}}_t^s$) as the completion of \mathcal{F}_t^s ($\bar{\mathcal{F}}_t^s$) in \mathcal{G} with respect to \mathcal{P} .

We will say that a family $A = \{A_{s,t}, 0 \leq s \leq t \leq T\}$ of random variables is an additive functional (AF) of \mathbb{X} if $A_{s,t}$ is \mathcal{G}_t^s -measurable for every $0 \leq s \leq t \leq T$ and $P_{s,x}(A_{s,t} = A_{s,u} + A_{u,t}, s \leq u \leq t \leq T) = 1$ for a.e. $(s, x) \in Q_{\hat{T}}$. If, in addition, $A_{s,\cdot}$ has $P_{s,x}$ -almost all continuous trajectories for a.e. $(s, x) \in Q_{\hat{T}}$, then A is called a continuous AF (CAF), and if $A_{s,\cdot}$ is an increasing process under $P_{s,x}$ for a.e. $(s, x) \in Q_{\hat{T}}$, it is called an increasing AF. If M is an AF such that for a.e. $(s, x) \in Q_{\hat{T}}$, $E_{s,x}|M_{s,t}|^2 < \infty$ and $E_{s,x}M_{s,t} = 0$ for $t \in [s, T]$ it is called a martingale AF (MAF). Finally, we say that A is an AF (CAF, increasing AF, MAF) in the strict sense if the corresponding property holds for every $(s, x) \in Q_{\hat{T}}$.

Now we recall some known facts about functionals in $\mathbb{L}_2(0, T; H_\varrho^{-1})$ (for details see, e.g., [8], [12]). Here and in what follows we will assume that $\varrho(x) = (1 + |x|^2)^{-\alpha}$, $x \in \mathbb{R}^d$, for some $\alpha \geq 0$ and $\int_{\mathbb{R}^d} \varrho(x) dx < \infty$.

It is known that if $\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1})$ then $\mu = f - \operatorname{div} \bar{f}$ for some $f, \bar{f} = (\bar{f}_1, \dots, \bar{f}_d) \in \mathbb{L}_{2,\varrho}(Q_T)$, i.e.

$$\Phi(\eta) = \langle f, \eta \rangle_{2,\varrho,T} + \langle \bar{f}, \nabla \eta \rangle_{2,\varrho,T}, \quad \eta \in \mathbb{L}_2(0, T; H_\varrho^1). \quad (4.1)$$

Let $\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1})$. We say that $\Phi \in \mathcal{M}^+(Q_T)$ if there is a measure $\mu \in \mathcal{M}^+(Q_T)$ such that

$$\Phi(\eta) = \int_{Q_T} \eta d\mu \quad (4.2)$$

for every $\eta \in C_0^\infty(Q_T)$. Let us note that the measure μ has the property that $\mu(\{t\} \times \mathbb{R}^d) = 0$ for every $t \in [0, T]$ (see [12]).

Let us consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $\mathbb{L}_{2,\varrho}(Q_T)$ with $\mathcal{F} = \mathbb{L}_2(0, T; H_\varrho^1)$ defined by the formula

$$\mathcal{E}(u, v) = \langle \nabla u, \nabla v \rangle_{2,\varrho,T}, \quad u, v \in \mathcal{F}.$$

It is easy to check that \mathcal{E} is regular and $C_0^\infty(Q_T)$ is its core. With the form $(\mathcal{E}, \mathcal{F})$ we may associate a Choquet capacity $\operatorname{Cap} : 2^{Q_T} \rightarrow [0, \infty]$ as follows. Let \mathcal{O} denote the family of all open subsets of Q_T . For $A \in \mathcal{O}$ we put

$$\operatorname{Cap}(A) = \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u, u),$$

where $\mathcal{L}_A = \{u \in \mathcal{F}; u \geq 1 \text{ a.e. on } A\}$ and $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \langle u, v \rangle_{2,\varrho,T}$. For $A \subset Q_T$ we put

$$\operatorname{Cap}(A) = \inf_{B \in \mathcal{O}, A \subset B} \operatorname{Cap}(B).$$

By [9, Theorem 2.1.5], for every $A \subset Q_T$ there exists a unique $e_A \in \bar{\mathcal{L}}_A = \{u \in \mathcal{F}; u \geq 1 \text{ Cap-q.e. on } A\}$ such that $\operatorname{Cap}(B) = \mathcal{E}_1(e_A, e_A)$. Since every functional in $\mathbb{L}_2(0, T; H_\varrho^{-1})$ is of the form (4.1), it follows that the measure μ is of finite energy integral (see Section 2.2 in [9]) and, by [9, Lemma 2.2.3], $\mu \ll \operatorname{Cap}$. Moreover, since every $\eta \in \mathcal{F}$ has Cap-quasi continuous version, repeating arguments of the proof of [9, Theorem 2.2.2] we can extend formula (4.2) to all $\eta \in \mathcal{F}$. In particular, given $\alpha \in \mathcal{B}_b(Q_T)$ and $\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}^+(Q_T)$ we may define $\alpha\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1})$ by the formula

$$\alpha\Phi(f) = \Phi(\alpha f) = \int_{Q_T} \alpha f d\mu, \quad f \in \mathbb{L}_2(0, T; H_\varrho^1),$$

where μ is the measure associated with Φ in the sense of (4.2).

Let us now consider parabolic capacity naturally related to the space \mathcal{W}_ϱ . We define the parabolic capacity of the Borel set $B \subset Q_{\hat{T}}$ as follows

$$\text{cap}(B) = P_m(\{\omega : (t, X_t(\omega)) \in B \text{ for some } t \in [0, T]\}),$$

where m is the Lebesgue measure on Q_T and $P_m(\Gamma) = \int_{Q_{\hat{T}}} P_{s,x}(\Gamma) ds dx$ for $\Gamma \in \mathcal{G}$. We say that $u \in \mathcal{B}(Q_T)$ is cap-quasi continuous if $[s, T] \ni t \mapsto u(t, X_t)$ is a $P_{s,x}$ -a.s. continuous process for a.e. $(s, x) \in Q_T$. It is known (see [12, 19]) that every $\eta \in \mathcal{W}_\varrho$ has a cap-quasi continuous version. In what follows we will always consider cap-quasi continuous versions of elements of \mathcal{W}_ϱ .

From [9, Theorem 2.1.4] it follows that if u, \bar{u} are Cap-quasi continuous and $u = \bar{u}$ a.e. then they are equal Cap-quasi everywhere. The same property hold for parabolic capacity.

Proposition 4.1. *If $u, \bar{u} \in \mathbb{L}_{2,\varrho}(Q_T)$ are cap-quasi continuous and $u = \bar{u}$ a.e. then $u = \bar{u}$ cap-quasi everywhere.*

Proof. Suppose that $\text{cap}(\{u \neq \bar{u}\}) > 0$. Then there exists $A \subset Q_{\hat{T}}$ such that $m(A) > 0$ and for every $(s, x) \in A$,

$$P_{s,x}(\{\omega : (t, X_t) \in \{u \neq \bar{u}\} \text{ for some } t \in [s, T]\}) > 0$$

Since the processes $t \mapsto u(t, X_t)$, $t \mapsto \bar{u}(t, X_t)$ have continuous trajectories,

$$0 < E_{s,x} \int_s^T |u - \bar{u}|^2(t, X_t) dt = \int_s^T \int_{\mathbb{R}^d} |u - \bar{u}|^2 p(s, x, \theta, y) d\theta dy. \quad (4.3)$$

Therefore, since $m(A) > 0$,

$$\begin{aligned} 0 &< \int_0^T \int_{\mathbb{R}^d} \left(E_{s,x} \int_s^T |u - \bar{u}|^2(t, X_t) dt \right) \varrho^2(x) ds \\ &\leq C \int_0^T \int_{\mathbb{R}^d} |u - \bar{u}|^2(\theta, y) \varrho^2(y) d\theta dy, \end{aligned}$$

the last inequality being a consequence of [12, Proposition 4.1]. Since (4.3) contradicts the assumption that $u = \bar{u}$ a.e., the proposition is proved. \square

Remark 4.2. If $\eta \in \mathcal{W}_\varrho$ then from [6, Appendix A.2] it follows that there exists $\{\eta_n\} \subset C_0^\infty(Q_T)$ such that $\eta_n \rightarrow \eta$ in \mathcal{W}_ϱ . By [12, Corollary 3.4] there exists a subsequence (still denoted by $\{n\}$) such that $\eta_n \rightarrow \bar{\eta}$ cap-q.e., where $\bar{\eta}$ is cap-quasi continuous version of η . On the other hand, $\eta_n \rightarrow \eta$ in \mathcal{E}_1 so by [9, Theorem 2.1.4] there exists a subsequence (still denoted by $\{n\}$) such that $\eta_n \rightarrow \tilde{\eta}$ Cap-q.e., where $\tilde{\eta}$ is Cap-quasi continuous version of η . From this we conclude that $\int \bar{\eta} d\mu = \int \tilde{\eta} d\mu$ for $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}(Q_T)$.

Let μ be a positive Radon measure on Q_T and let K be an increasing CAF of \mathbb{X} . We will say that μ corresponds to K or K corresponds to μ (and write $\mu \sim K$) if

$$E_{s,x} \int_s^T \eta(t, X_t) dK_{s,t} = \int_{Q_{sT}} \eta(t, y) p(s, x, t, y) d\mu(t, y) \quad (4.4)$$

for every $\eta \in \mathcal{B}^+(Q_T)$ and a.e. $(s, x) \in Q_T$.

Observe that if μ corresponds to some increasing CAF of \mathbb{X} , then $\mu \ll \text{cap}$ since $p > 0$. Note also that from [12, Corollary 3.5] it follows that every $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}^+(Q_T)$ has a corresponding CAF of \mathbb{X} .

Now we prove some properties of the Laplace transform of time-inhomogeneous additive functionals. Analogous properties for time-homogeneous additive functionals are to be found for instance in [23, Chapter X].

Let A be an additive functional of \mathbb{X} and let $\alpha \geq 0$. The function

$$U_A^\alpha(s, x) = E_{s,x} \int_s^T e^{-\alpha(t-s)} dA_{s,t}, \quad (s, x) \in Q_T$$

is called the Laplace transform of the AF A or the α -potential of A . If $f \in \mathcal{B}_b(Q_T)$ and $f \cdot A$ is the functional defined by $(f \cdot A)_{s,t} = \int_s^t f(\theta, X_\theta) d\theta$, $0 \leq s \leq t \leq T$, then $U_A^\alpha f$ denotes the α -potential of $f \cdot A$, that is

$$U_A^\alpha f(s, x) = E_{s,x} \int_s^T e^{-\alpha(t-s)} f(t, X_t) dA_{s,t}, \quad (s, x) \in Q_T.$$

If $A_{s,t} = t - s$, then we denote $U_A^\alpha f$ by $U^\alpha f$.

Lemma 4.3. *For any additive functional A and any $f \in \mathcal{B}_b(Q_T)$,*

$$U_A^\alpha(U^\alpha f)(s, x) = E_{s,x} \int_s^T e^{-\alpha(t-s)} f(t, X_t) dA_{s,t} dt$$

for almost every $(s, x) \in \check{Q}_T$.

Proof. By the definitions of α -potential the fact that $(X, P_{s,x})$ is a Markov process and Fubini's theorem,

$$\begin{aligned} U_K^\alpha(U^\alpha f)(s, x) &= E_{s,x} \int_s^T e^{-\alpha(t-s)} \left(E_{t,X_t} \int_t^T e^{-\alpha(\theta-s)} f(\theta, X_\theta) d\theta \right) dA_{s,t} \\ &= E_{s,x} \int_s^T e^{-\alpha(t-s)} E_{s,x} \left(\int_t^T e^{-\alpha(\theta-s)} f(\theta, X_\theta) d\theta | \mathcal{G}_t^s \right) dA_{s,t} \\ &= E_{s,x} \int_s^T \int_t^T e^{-\alpha(\theta-s)} f(\theta, X_\theta) dA_{s,\theta} d\theta. \end{aligned}$$

□

Proposition 4.4. *Let μ_1, μ_2 be Radon measures such that there exist additive functionals K, L such that $\mu_1 \sim K$, $\mu_2 \sim L$. If $\mu_1 \leq \mu_2$ then $K \leq L$ in the sense that $K_{t',t} \leq L_{t',t}$ for every $s \leq t' \leq t \leq T$, $P_{s,x}$ -a.s. for a.e. $(s, x) \in \check{Q}_T$.*

Proof. By the assumptions,

$$U_K^\alpha f \leq U_L^\alpha f, \quad \alpha \geq 0$$

for every $f \in C_0^+(\check{Q}_T)$. Using the theorem on monotone classes one can show that the above inequalities holds for any $f \in \mathcal{B}_b^+(Q_T)$. In particular, for any $f \in C_0^+(\check{Q}_T)$ and $\alpha \geq 0$,

$$U_K^\alpha U^\alpha f \leq U_L^\alpha U^\alpha f.$$

From this and Lemma 4.3 we conclude that for a.e $(s, x) \in \check{Q}_T$,

$$E_{s,x}f(t, X_t)K_{s,t} \leq E_{s,x}f(t, X_t)L_{s,t}, \quad t \in [s, T]$$

for every $f \in C_0^+(\check{Q}_T)$. Suppose that $s \leq s' \leq t' \leq t$. By the above, additivity of K, L and the Markov property,

$$\begin{aligned} E_{s,x}f(t', X_{t'})K_{s',t} &= E_{s,x}f(t', X_{t'})K_{s',t'} + E_{s,x}f(t', X_{t'})K_{t',t} \\ &= E_{s,x}(E_{s',X_{s'}}(f(t', X_{t'})K_{s',t'})) + E_{s,x}(f(t', X_{t'})E_{t',X_{t'}}K_{t',t}) \\ &\leq E_{s,x}(E_{s',X_{s'}}(f(t', X_{t'})L_{s',t'})) + E_{s,x}(f(t', X_{t'})E_{t',X_{t'}}(L_{t',t})) \\ &= E_{s,x}f(t', X_{t'})L_{s',t}. \end{aligned}$$

By induction, for every $0 \leq t' \leq t_1 \leq \dots \leq t_k \leq t \leq T$ we have

$$E_{s,x} \prod_{i=1}^k f(t_i, X_{t_i})K_{t_i,t} = E_{s,x} \prod_{i=1}^k f(t_i, X_{t_i})L_{t_i,t},$$

from which the lemma follows. \square

Corollary 4.5. *If $\mu \sim K$, $\mu \sim L$ then $K = L$.*

It is known (see [13, 25]) that there exist CAF A in the strict sense and a continuous MAF M in the strict sense such that

$$X_t - X_s = M_{s,t} + A_{s,t}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.},$$

for every $(s, x) \in Q_{\hat{T}}$, and moreover, $M_{s,\cdot}$ is a $(\{\mathcal{G}_t^s\}, P_{s,x})$ -square-integrable martingale on $[s, T]$ with the co-variation given by

$$\langle M_{s,\cdot}^i, M_{s,\cdot}^j \rangle_t = \int_s^t a_{ij}(\theta, X_\theta) d\theta, \quad t \in [s, T], \quad i, j = 1, \dots, d, \quad (4.5)$$

while $A_{s,\cdot}$ is a process of $P_{s,x}$ -zero-quadratic variation on $[0, T]$. In particular, $X_\cdot - X_s$ is a $(\{\mathcal{G}_t^s\}, P_{s,x})$ -Dirichlet process in the sense of Föllmer.

Observe that by (4.5),

$$B_{s,t} = \int_s^t \sigma^{-1}(\theta, X_\theta) dM_{s,\theta}, \quad t \in [s, T]$$

is a $(\{\mathcal{G}_t^s\}, P_{s,x})$ -Wiener process. In [13] it is proved that it has the representation property. Therefore existence and uniqueness of solutions of (1.9) follows from known results for usual BSDEs (see [7]), and moreover, we may apply Theorem 3.2 to RBSDEs with the Wiener process $B_{s,\cdot}$ defined on the stochastic basis $(\Omega, \mathcal{G}, \{\mathcal{G}_t^s\}, P_{s,x})$.

We say that a pair $(Y^{s,x}, Z^{s,x})$ of $\{\mathcal{G}_t^s\}$ -adapted processes on $[s, T]$ is a solution of GBSDE (1.5) if $E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x}|^2 < \infty$, $E_{s,x} \int_s^T |Z_t^{s,x}|^2 dt < \infty$ and (1.5) is satisfied $P_{s,x}$ -a.s.

Let $S^{s,x}$ be a continuous $\{\mathcal{G}_t^s\}$ adapted process. A triple $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of $\{\mathcal{G}_t^s\}$ -adapted process on $[s, T]$ is a solution of RBSDE (1.9) if $E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x}|^2 < \infty$, $E_{s,x} \int_s^T |Z_t^{s,x}|^2 dt < \infty$, $E_{s,x} |K_T^{s,x}|^2 < \infty$ and (1.9) is satisfied $P_{s,x}$ -a.s.

In the rest of this section we assume that

- (H1) $\varphi \in \mathbb{L}_{2,\varrho}(\mathbb{R}^d)$,
- (H2) There exist $M > 0$, $\gamma \in \mathbb{L}_{2,\varrho}(Q_T)$ such that $|f(t, x, y, z)| \leq |\gamma(t, x)| + M(|y| + |z|)$ for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,
- (H3) $f(t, x, \cdot, \cdot)$ is continuous for a.e. $(t, x) \in Q_T$,
- (H4) $|g(t, x, y)| \leq M$ for some $M > 0$ and $g(t, x, \cdot)$ is continuous for every $(t, x) \in Q_T$.

Let $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}^+(Q_T)$. We say that $u \in \mathcal{W}_\varrho$ is a strong solution of the problem (1.1) if $u(T) = \varphi$ in $\mathbb{L}_{2,\varrho}(\mathbb{R}^d)$ and

$$\left\langle \frac{\partial u}{\partial t}, \eta \right\rangle_{\varrho, T} + \langle L_t u, \eta \rangle_{\varrho, T} = -\langle f_u, \eta \rangle_{2, \varrho, T} - \int_{Q_T} \eta g(u) d\mu$$

for every $\eta \in C_0^\infty(Q_T)$.

Notice that the terminal condition in the above definition is meaningful since it is known that $\mathcal{W}_\varrho \subset C([0, T], \mathbb{L}_{2,\varrho}(\mathbb{R}^d))$.

The following theorem has been proved in [12] (see [12, Corollary 3.3]).

Theorem 4.6. *Let $h \in \mathcal{W}_\varrho$. If $\frac{\partial h}{\partial t} + L_t h = \Phi$ and $\Phi = \alpha_1 \Phi_1 - \alpha_2 \Phi_2$, where $\Phi_1, \Phi_2 \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, $\Phi_1, \Phi_2 \geq 0$, $\alpha_1, \alpha_2 \in \mathcal{B}_b(Q_T)$, then there exist a cap-quasi continuous version of h , still denoted by h , and square-integrable increasing CAFs C, R such that*

$$\begin{aligned} h(t, X_t) &= h(T, X_T) - \int_t^T \alpha_1(\theta, X_\theta) dC_{s,\theta} + \int_t^T \alpha_2(\theta, X_\theta) dR_{s,\theta} \\ &\quad - \int_t^T \sigma \nabla h(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \end{aligned}$$

for a.e. $(s, x) \in Q_{\hat{T}}$, and if μ_1, μ_2 are Radon measures associated with Φ_1 and Φ_2 , respectively, then for a.e. $(s, x) \in Q_{\hat{T}}$,

$$E_{s,x} \int_s^T \xi(\theta, X_\theta) dC_{s,\theta} = \int_s^T \int_{\mathbb{R}^d} \xi(\theta, y) p(s, x, \theta, y) d\mu_1(\theta, y), \quad (4.6)$$

$$E_{s,x} \int_s^T \xi(\theta, X_\theta) dR_{s,\theta} = \int_s^T \int_{\mathbb{R}^d} \xi(\theta, y) p(s, x, \theta, y) d\mu_2(\theta, y) \quad (4.7)$$

for every $\xi \in C_0(Q_T)$ and a.e. $(s, x) \in Q_T$.

The above theorem will be used in the proof of the following theorem on existence and stochastic representation of strong solutions of (1.1) and will play key role in the proof of Theorem 5.2 on existence, approximation and stochastic representation of strong solutions of the obstacle problem (1.8).

Theorem 4.7. *Assume that (H1)–(H4) are satisfied and $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}^+(Q_T)$. Then there exists a minimal strong solution $u \in \mathcal{W}_\varrho$ of the problem (1.1). Moreover, the pair $(u(t, X_t), \sigma \nabla u(t, X_t))$, $t \in [s, T]$ is a minimal solution of the GBSDE*

$$\begin{aligned} u(t, X_t) &= \varphi(X_T) + \int_t^T f_u(\theta, X_\theta) d\theta + \int_t^T g(u)(\theta, X_\theta) dR_{s,\theta} \\ &\quad - \int_t^T \sigma \nabla u(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \end{aligned}$$

where $\mu \sim R$.

Proof. First we assume additionally that f is Lipschitz continuous with respect to x, y uniformly in t . Let $(Y^{s,x}, Z^{s,x})$ be a solution of (1.5). Existence follows from Proposition 2.6 as the assumptions of this theorem are satisfied for a.e. $(s, x) \in Q_T$ (see Corollary 3.5 and Proposition 4.1 in [12]). Let $g_M(u) = g(u) + M$ so that $g_M(u) \geq 0$, and let

$$K_{s,t}^{s,x} = \int_s^t g_M(u)(\theta, X_\theta, Y_\theta^{s,x}) dR_{s,\theta}, \quad t \in [s, T].$$

Then we can write (1.5) in the form

$$Y_t^{s,x} = \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x}) + K_T^{s,x} - K_t^{s,x} - MR_{T,t} - \int_t^T Z_\theta^{s,x} dB_{s,\theta}.$$

Let $(Y^{s,x,n}, Z^{s,x,n})$ be a solution of the BSDE

$$\begin{aligned} Y_t^{s,x,n} &= \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x,n}, Z_\theta^{s,x,n}) + K_T^{s,x,n} - K_t^{s,x,n} - MR_{T,t} \\ &\quad - \int_t^T Z_\theta^{s,x,n} dB_{s,\theta}, \quad t \in [s, T], \end{aligned}$$

where $K_{s,t}^{s,x,n} = \int_s^t n(Y_\theta^{s,x,n} - Y_\theta^{s,x})^- d\theta$. In much the same way as in the proof of the approximation result in [7, Section 6] (see also [21]) one can show that

$$\begin{aligned} E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x,n} - Y_t^{s,x}|^2 + E_{s,x} \int_s^T |Z_t^{s,x,n} - Z_t^{s,x}|^2 dt \\ + E_{s,x} \sup_{s \leq t \leq T} |K_{s,t}^{s,x,n} - K_{s,t}^{s,x}|^2 \rightarrow 0 \end{aligned} \quad (4.8)$$

as $n, m \rightarrow \infty$, and

$$\begin{aligned} E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x,n}|^2 + E_{s,x} \int_s^T |Z_t^{s,x,n}|^2 dt + E_{s,x} |K_{s,T}^{s,x,n}|^2 \\ \leq CE_{s,x} (|\varphi(X_T)|^2) + \int_s^T |\gamma(t, X_t)|^2 dt + \sup_{s \leq t \leq T} |Y_t^{s,x}|^2 + |R_{s,T}|^2. \end{aligned} \quad (4.9)$$

Let us observe now that defining

$$\tilde{u}(s, x) = E_{s,x} \left(\varphi(X_T) + \int_s^T f(t, X_t, Y_t^{s,x}, Z_t^{s,x}) dt + \int_s^T g(t, X_t, Y_t^{s,x}) dR_{s,\theta} \right)$$

we get using the Markov property of \mathbb{X} that

$$\begin{aligned} Y_t^{s,x} &= E_{s,x}(Y_t^{s,x} | \mathcal{G}_t^s) \\ &= E_{s,x} \left(\varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x}) d\theta + \int_t^T g(\theta, X_\theta, Y_\theta^{s,x}) dR_{s,\theta} | \mathcal{G}_t^s \right) \\ &= E_{t,X_t} \left(\varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x}) d\theta + \int_t^T g(\theta, X_\theta, Y_\theta^{s,x}) dR_{t,\theta} \right) \\ &= u(t, X_t). \end{aligned} \quad (4.10)$$

$P_{s,x}$ -a.s. for every $t \in [s, T]$. Hence,

$$\begin{aligned} Y_t^{s,x,n} &= \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x,n}, Z_\theta^{s,x,n}) d\theta \\ &\quad + \int_t^T n(Y_\theta^{s,x,n} - \tilde{u}(\theta, X_\theta))^- d\theta - MR_{T,t} - \int_t^T Z_\theta^{s,x,n} dB_{s,\theta} \end{aligned} \quad (4.11)$$

From (4.10), Corollary 3.5 and Proposition 4.1 in [12] it follows that $\tilde{u} \in \mathbb{L}_{2,\varrho}(Q_T)$. In [6] it is proved that there exists a unique strong solution u_n of the problem

$$\frac{\partial u_n}{\partial t} + L_t u_n = -f_{u_n} - n(u_n - \tilde{u})^- + M\mu, \quad u_n(T) = \varphi, \quad (4.12)$$

while from Theorem 4.6 it follows that there is a cap-quasi continuous version of u_n (still denoted by u_n) such that the pair $(u_n(t, X_t), \sigma \nabla u_n(t, X_t))$, $t \in [s, T]$, is a solution of (4.11). Since (4.11) has a unique solution,

$$Y_t^{s,x,n} = u_n(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad Z_t^{s,x,n} = \sigma \nabla u_n(t, X_t), \quad \lambda \otimes P_{s,x}\text{-a.s.} \quad (4.13)$$

In view of the above we may consider versions of $Y^{s,x,n}$, $Z^{s,x,n}$, $Y^{s,x}$, $Z^{s,x}$ which do not depend on s, x . Furthermore, from (4.10) it follows that $K^{s,x,n}$, $K^{s,x}$ have versions not depending on s, x . In what follows we consider versions of the processes not depending on s, x , and consequently, we drop the superscript s, x in the notation. Write $d\nu_n = n(u_n - \tilde{u})^- dm$. From (4.8), (4.9) and [12, Proposition 4.1] it follows that

$$\|\nabla u_n - \nabla u_m\|_{2\varrho, T}^2 \leq C \int_{Q_T} \left(E_{s,x} \int_s^T |Z_\theta^n - Z_\theta^m|^2 d\theta \right) \varrho^2(x) ds dx \rightarrow 0$$

and

$$\|u_n - u_m\|_{2\varrho, T}^2 \leq C \int_{Q_T} \left(E_{s,x} \int_s^T |Y_\theta^n - Y_\theta^m|^2 d\theta \right) \varrho^2(x) ds dx \rightarrow 0.$$

Now, if we set $u(t, x) = \lim_{n \rightarrow \infty} u_n(t, x)$ if the limit exists and is finite and $u(s, x) = 0$ otherwise, then by the above, $u_n \rightarrow u$ in $\mathbb{L}_2(0, T; H_\varrho^1)$ and

$$E_{s,x} \sup_{s \leq t \leq T} |u_n(t, X_t) - u(t, X_t)|^2 \rightarrow 0$$

for a.e. $(s, x) \in Q_{\hat{T}}$ which shows that u is cap-quasi continuous. Now, let $\eta \in C_0^\infty(Q_T)$. Then from the definition of the solution of (4.12),

$$\left\langle \frac{\partial u_n}{\partial t}, \eta \right\rangle_{\varrho, T} + \langle L_t u_n, \eta \rangle_{\varrho, T} = -\langle f_{u_n}, \eta \rangle_{2, \varrho, T} - \int_{Q_T} \eta d\nu_n + \int_{Q_T} \eta M d\mu. \quad (4.14)$$

Hence, by the integration by parts formula,

$$\begin{aligned} \langle u_n, \frac{\partial \eta}{\partial t} \rangle_{2, \varrho, T} - \langle L_t u_n, \eta \rangle_{\varrho, T} &= \langle \varphi, \eta(T) \rangle_{2, \varrho} - \langle u(0), \eta(0) \rangle_{2, \varrho} + \langle f_{u_n}, \eta \rangle_{2, \varrho, T} \\ &\quad + \int_{Q_T} \eta d\nu_n - \int_{Q_T} \eta M d\mu. \end{aligned} \quad (4.15)$$

By (4.9), Proposition 2.1, Proposition 4.1 and Corollary 3.5 in [12],

$$\sup_{0 \leq t \leq T} \|u_n(t)\|_{2,\varrho}^2 + \|\nabla u_n\|_{2,\varrho,T}^2 \leq C(\|\varphi\|_{2,\varrho}^2 + \|\gamma\|_{2,\varrho,T}^2 + \|\mu\|_{\mathbb{L}_2(0,T;H_\varrho^{-1})}). \quad (4.16)$$

Using this one can check easily that $\{\nu_n\}$ is tight. Therefore without loss of generality we may and will assume that $\{\nu_n\}$ converges weakly to some measure ν . Consequently, letting $n \rightarrow \infty$ in (4.14) we conclude that there is functional Ψ on $C_0^\infty(\check{Q}_T)$ such that

$$\Psi(\eta) + \langle L_t u, \eta \rangle_{\varrho,T} = -\langle f_u, \eta \rangle_{2,\varrho,T} - \int_{Q_T} \eta d\nu + \int_{Q_T} \eta M d\mu. \quad (4.17)$$

We know that $\nu_n \sim K^n$, i.e. for a.e. $(s, x) \in Q_T$,

$$E_{s,x} \int_s^T \eta(\theta, X_\theta) dK_{s,\theta}^n = \int_s^T \int_{\mathbb{R}^d} \eta(\theta, y) p(s, x, \theta, y) d\nu_n(\theta, y)$$

for every $\eta \in C_0(Q_T)$. Hence, by (4.8), for a.e. $(s, x) \in Q_T$,

$$E_{s,x} \int_s^T \eta(\theta, X_\theta) dK_{s,\theta} = \int_s^T \int_{\mathbb{R}^d} \eta(\theta, y) p(s, x, \theta, y) d\nu(\theta, y)$$

for every $\eta \in C_0(Q_T)$. Therefore $\nu \sim K$. On the other hand, by the definition of K^n ,

$$E_{s,x} \int_s^T \eta(\theta, X_\theta) dK_{s,\theta}^n = E_{s,x} \int_s^T \eta(\theta, X_\theta) g_M(u_n)(\theta, X_\theta) dR_{s,\theta}$$

for $\eta \in C_0(Q_T)$ and a.e. $(s, x) \in Q_T$. Using this and (4.8), (4.9) we conclude that $g_M(u) d\mu \sim K$. By uniqueness, $d\nu = g_M(u) d\mu$. Thus, (4.17) takes the form

$$\Psi(\eta) + \langle L_t u, \eta \rangle_{\varrho,T} = -\langle f_u, \eta \rangle_{2,\varrho,T} - \int_{Q_T} \eta g(u) d\mu.$$

Since $g(u) d\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, using arguments following (4.2) we can extend Ψ from $C_0^\infty(Q_T)$ to the functional $\bar{\Psi}$ on $\mathbb{L}_2(0, T; H_\varrho^{-1})$. Moreover, passing to the limit in (4.15) and subtracting (4.17) we see that

$$\Psi(\eta) = -\langle u, \frac{\partial \eta}{\partial t} \rangle_{2,\varrho,T}$$

for every $\eta \in C_0^\infty(\check{Q}_T)$. Therefore $\bar{\Psi} = \frac{\partial u}{\partial t}$ and $u \in \mathcal{W}_\varrho$. Thus, u is a solution of the problem (1.1) and, by (4.8)–(4.13), the pair $(u(t, X_t), \sigma \nabla u(t, X_t))$, $t \in [s, T]$, is a solution of GBSDE (1.5). Existence of the minimal solution of (1.1) follows now from existence of the minimal solution of (1.5) (see Proposition 2.6).

We now show how to dispense with the assumption that f is Lipschitz continuous. Let $\{f^n\}$ be the sequence of approximations of f considered in Lemma 2.4. Let $(Y^{s,x,n}, Z^{s,x,n}) = (Y^n, Z^n)$ be a minimal solution of (1.5) with f replaced by f^n and let $u_n \in \mathcal{W}_\varrho$ be a solution of the problem

$$\frac{\partial u_n}{\partial t} + L_t u_n = -f_{u_n}^n - g(u_n) \mu, \quad u_n(T) = \varphi. \quad (4.18)$$

From the first part of the proof we know that (4.13) is satisfied. Furthermore, arguing as in the case of usual BSDEs (see [14]) one can show that

$$E_{s,x} \sup_{s \leq t \leq T} |Y_t^n - Y_t^m|^2 + E_{s,x} \int_s^T |Z_t^n - Z_t^m|^2 dt \rightarrow 0 \quad (4.19)$$

as $n, m \rightarrow \infty$ and

$$\begin{aligned} & E_{s,x} \left(\sup_{s \leq t \leq T} |Y_t^n|^2 + \int_s^T |Z_t^n|^2 dt \right) \\ & \leq C E_{s,x} \left(|\varphi(X_T)|^2 + \int_s^T |\gamma(t, X_t)|^2 dt + |R_{s,T}|^2 \right). \end{aligned} \quad (4.20)$$

Set $u(t, x) = \lim u_n(t, x)$ if the limit exists and is finite and $u(s, x) = 0$ otherwise. As in the first part of the proof we conclude from (4.19), (4.20) and Proposition 4.1 and Corollary 3.5 in [12] that $u_n \rightarrow u$ in $\mathbb{L}_2(0, T; H_\varrho^{-1})$ and u is cap-quasi continuous. By the definition of the solution of (4.18),

$$\left\langle \frac{\partial u_n}{\partial t}, \eta \right\rangle_{\varrho, T} + \langle L_t u_n, \eta \rangle_{\varrho, T} = -\langle f_{u_n}^n, \eta \rangle_{2, \varrho, T} - \int_{Q_T} \eta g(u_n) d\mu \quad (4.21)$$

for every $\eta \in C_0^\infty(Q_T)$. The above equality may be extended to all $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$ (see comments following (4.2)). Moreover, taking u_n as a test function in (4.21) and using the properties of the approximating sequence $\{f^n\}$ we conclude that (4.16) is satisfied and for every $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$,

$$\sup_{n \geq 1} \left| \left\langle \frac{\partial u_n}{\partial t}, \eta \right\rangle_{\varrho, T} \right| < \infty.$$

This proves that $u \in \mathcal{W}_\varrho$ and $\frac{\partial u_n}{\partial t} \rightarrow \frac{\partial u}{\partial t}$ weakly in \mathcal{W}'_ϱ . By (4.8), $u_n \rightarrow u$ cap-quasi everywhere. Hence letting $n \rightarrow \infty$ in (4.21) shows that u is a solution of (1.1).

Suppose that $v \in \mathcal{W}_\varrho$ is another solution of (1.1). Then by Theorem 4.6 the pair $(v(t, X_t), \sigma \nabla v(t, X_t))$, $t \in [s, T]$ is a solution of GBSDE (1.5) for a.e. $(s, x) \in Q_{\hat{T}}$. On the other hand, arguing as in the proof of Proposition 2.6 one can show that $(u(t, X_t), \sigma \nabla u(t, X_t))$, $t \in [s, T]$, is the minimal solution of (1.5) for a.e. $(s, x) \in Q_{\hat{T}}$. This implies that $u(t, X_t) \leq v(t, X_t)$, $t \in [s, T]$ for a.e. $(s, x) \in Q_{\hat{T}}$ which is equivalent to the fact that $u \leq v$ cap-quasi everywhere. Thus, u is the minimal solution of (1.1), and the proof is complete. \square

Remark 4.8. From Theorem 4.7 and Remark 2.7 it follows that under assumptions of Theorem 4.7 there exists a maximal solution $u \in \mathcal{W}_\varrho$ of (1.1).

Note that the stochastic representation of weak solutions of the problem (1.1) with $g = 0$ was obtained in [26].

5 Stochastic representation of solutions of the obstacle problem

In this section we consider stochastic homographic approximation for RBSDEs in a Markovian framework. We assume that the final condition ξ , coefficient f and obstacle

S are explicit functionals of a diffusion associated with the divergence form operator L_t defined by (1.2).

We will need the following additional hypotheses.

(H5) There is $L > 0$ such that $|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

(H6) $h \in \mathcal{W}_\varrho$, $\varphi(x) \geq h(T, x)$ for a.e. $x \in \mathbb{R}^d$.

We say that $\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}(Q_T)$ if $\Phi \in \mathbb{L}_2(0, T; H_\varrho^{-1})$ and there exists $\mu \in \mathcal{M}(Q_T)$ such that (4.2) is satisfied for every $\eta \in C_0^\infty(Q_T)$.

Proposition 5.1. *Let $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}(Q_T)$ and let $\mu^+ - \mu^-$ be the Jordan decomposition of μ . Then $\mu^+, \mu^- \in \mathbb{L}_2(0, T; H_\varrho^{-1}) \cap \mathcal{M}^+(Q_T)$.*

Proof. Step 1. First we show that $\mu^+, \mu^- \ll \text{Cap}$. Without lost of generality we can assume that $\text{supp}[\mu] \subset\subset Q_T$. Let $X = \text{supp}[\mu]$ and $X = A \cup B$, where $A, B \in \mathcal{B}(Q_T)$ are from Hahn's decomposition of signed measure. Let $A_\varepsilon, B_\varepsilon$ will be compact, $A_\varepsilon \subset A, B_\varepsilon \subset B$ and $|\mu|(A - A_\varepsilon) < \varepsilon$, $|\mu|(B - B_\varepsilon) < \varepsilon$. Let $K \subset\subset A_\varepsilon$ and $\text{Cap}(K) = 0$. We will show that $\mu(K) = 0$. Since $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, $\mu = f - \text{div}(\bar{f})$ for some $f, \bar{f} \in \mathbb{L}_{2,\varrho}(Q_T)$. Let A_ε^δ will be an open subset of X such that $A_\varepsilon \subset A_\varepsilon^\delta$ and $|\mu|(A_\varepsilon^\delta - A_\varepsilon) < \delta$. Let $\xi \in C_0^\infty(A_\varepsilon^\delta)$, $\xi|_{A_\varepsilon} = 1, \xi \geq 0$. So $\mu^\xi := \xi d\mu = f^\xi - \text{div}(\bar{f}^\xi)$ for some $f^\xi, \bar{f}^\xi \in \mathbb{L}_{2,\varrho}(Q_T)$. For $\eta \in C_0^\infty(Q_T)$ such that $\mathbf{1}_K \leq \eta \leq 2$ we have

$$\begin{aligned} \mu^\xi(K) &= \int_{Q_T} \mathbf{1}_K d\mu^\xi \leq \int_{Q_T} \eta \mathbf{1}_{A_\varepsilon} d\mu^\xi = \int_{Q_T} \eta d\mu^\xi + \int_{Q_T} (\eta \mathbf{1}_{A_\varepsilon} - \eta) d\mu^\xi \\ &\leq \int_{Q_T} \eta d\mu^\xi + 2|\mu|(A_\varepsilon^\delta - A_\varepsilon) \leq C(f^\xi, \bar{f}^\xi) \sqrt{\mathcal{E}_1(\eta, \eta)} + 2\delta. \end{aligned}$$

By the above and [9, Lemma 2.2.7] we get

$$\mu(K) = \mu^\xi(K) \leq C(f^\xi, \bar{f}^\xi) \text{Cap}(K) + 2\delta = 2\delta.$$

Because $\delta > 0$ was arbitrary we get that $\mu^\xi(K) = 0$ and hence that $\mu(K) = 0$. Similary we obtain that if $K \subset\subset B_\varepsilon$ and $\text{Cap}(K) = 0$ then $\mu(K) = 0$. Let $D \in \mathcal{B}(Q_T)$ and $\text{Cap}(D) = 0$. By [9, Theorem 2.1.4] the last statement is equivalent to $\text{Cap}(K) = 0$ for every compact $K \subset D$. Now we have that

$$\mu(K) = \mu(A \cap K) + \mu(B \cap K) = \lim_{\varepsilon \rightarrow 0} (\mu(A_\varepsilon \cap K) + \mu(B_\varepsilon \cap K)) = 0.$$

Therefore $\mu(D) = 0$. This shows that $\mu \ll \text{Cap}$ and as an immediate consequence that $\mu^+, \mu^- \ll \text{Cap}$.

Step 2. Let $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$. First assume additionally that η is bounded. Since \mathcal{E} is regular there is a sequence $\{\eta_n\} \subset C_0^\infty(Q_T)$ converging to η in \mathcal{E}_1 . By [9, Theorem 2.1.4] there exists subsequence $\{n_k\}$ such that $\eta_{n_k} \rightarrow \eta$ q.e.. We know that

$$\int_{Q_T} \eta_{n_k} d\mu = \langle f, \eta_{n_k} \rangle_{2,\varrho,T} + \langle \bar{f}, \nabla \eta_{n_k} \rangle_{2,\varrho,T}$$

so letting $k \rightarrow \infty$ and using the Lebesgue dominated convergence theorem we get

$$\int_{Q_T} \eta d\mu = \langle f, \eta \rangle_{2,\varrho,T} + \langle \bar{f}, \nabla \eta \rangle_{2,\varrho,T}. \quad (5.1)$$

Since for every $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$ and $c \in \mathbb{R}$ $\eta^+, \eta^-, \eta^+ \wedge c, \eta^- \wedge c \in \mathbb{L}_2(0, T; H_\varrho^1)$, using standard arguments we can show that (5.1) holds true for any $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$. In particular, it follows that $\int_{Q_T} \eta d|\mu| < \infty$ for $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$. Since $\mu = \mu^+ - \mu^- \in \mathbb{L}_2(0, T; H_\varrho^{-1})$ by the assumption, to prove that $\mu^+, \mu^- \in \mathbb{L}_2(0, T; H_\varrho^{-1})$ it suffices to show that $|\mu| = \mu^+ + \mu^- \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, that is that the functional $|\mu|$ is continuous. Since $\mathbb{L}_2(0, T; H_\varrho^1)$ is a Hilbert space, it follows from the closed graph theorem that to prove continuity of $|\mu|$ it suffices to show that the $|\mu|$ is closed. But the last property follows easily from [9, Theorem 2.1.4]. \square

We say that a pair (u, μ) , where μ is a positive Radon measure on Q_T and $u \in \mathcal{W}_\varrho$, is a strong solution of the obstacle problem (1.8) if

$$\left\langle \frac{\partial u}{\partial t}, \eta \right\rangle_{\varrho,T} + \langle L_t u, \eta \rangle_{\varrho,T} = \langle f_u, \eta \rangle_{2,\varrho,T} + \int_{Q_T} \eta d\mu \quad (5.2)$$

for any $\eta \in \mathcal{F}$, and

$$u(T) = \varphi, \quad u \geq h \text{ on } Q_T, \quad \int_{Q_T} (u - h) d\mu = 0. \quad (5.3)$$

It is easily seen that strong solution of an obstacle problem is a strong solution of (1.7) in the variational sense (for the definition of solution in the variational sense see [4, 5, 15]). Therefore from known results on uniqueness of variational problems it follows that under (H1), (H2), (H5), (H6) strong solution of (1.8) is unique. Let us observe also that from (5.2) it follows that $\mu \in \mathbb{L}_2(0, T; H_\varrho^{-1})$, which implies that the integral in (5.3) is well defined.

Theorem 5.2. *Assume that (H1), (H2), (H5), (H6) are satisfied and $\frac{\partial h}{\partial t} + L_t h \in \mathcal{M}(Q_T)$. Then there exists a strong solution (u, μ) of (1.8) and if u_n , $n \in \mathbb{N}$, is a maximal solution of the Cauchy problem*

$$\frac{\partial u_n}{\partial t} + L_t u_n = -f_{u_n} - \mu_n, \quad u_n(T) = \varphi \quad (5.4)$$

with

$$\mu_n = \frac{1}{1 + n|u_n - h|} \left(\frac{\partial h}{\partial t} + L_t h + f_h \right)^-$$

then

- (i) $u_n \geq h$, $u_n \searrow u$ a.e., $u_n \rightarrow u$ in $\mathbb{L}_2(0, T; H_\varrho^1) \cap C([0, T], \mathbb{L}_{2,\varrho}(\mathbb{R}^d))$,
- (ii) $\mu_n \Rightarrow \mu$ and $\mu_n \rightarrow \mu$ in \mathcal{W}'_ϱ .

Moreover for a.e. $(s, x) \in Q_{\hat{T}}$ there exists a solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of (1.9). In fact, for a.e. $(s, x) \in Q_{\hat{T}}$,

$$Y_t^{s,x} = \tilde{u}(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad Z_t^{s,x} = \sigma \nabla \tilde{u}(t, X_t), \quad \lambda \otimes P_{s,x}\text{-a.s.} \quad (5.5)$$

for some version \tilde{u} of u and

$$E_{s,x} \int_s^T \xi(t, X_t) dK_t^{s,x} = \int_s^T \int_{\mathbb{R}^d} \xi(t, y) p(s, x, t, y) d\mu(t, y) \quad (5.6)$$

for every $\xi \in C_0(Q_T)$ and a.e. $(s, x) \in Q_T$

Proof. Let $\Phi = \frac{\partial h}{\partial t} + L_t h + f_h$. By the assumptions on h , $\frac{\partial h}{\partial t} + L_t h = -f_h + \Phi^+ - \Phi^-$ for some functionals $\Phi^+, \Phi^- \in \mathbb{L}_2(0, T; H_\rho^{-1}) \cap \mathcal{M}^+(Q_T)$. Hence, by Theorem 4.6, there exist cap-quasi continuous versions of h and u_n (still denoted by h, u_n) such that for a.e. $(s, x) \in Q_{\hat{T}}$,

$$\begin{aligned} h(t, X_t) &= h(T, X_T) + \int_t^T f_h(\theta, X_\theta) d\theta - C_{t,T} + R_{t,T} \\ &\quad - \int_t^T \sigma \nabla h(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \end{aligned}$$

and

$$\begin{aligned} u_n(t, X_t) &= \varphi(X_T) + \int_t^T f_{u_n}(\theta, X_\theta) d\theta + \int_t^T \alpha_n(\theta, X_\theta) dR_{s,\theta} \\ &\quad - \int_t^T \sigma \nabla u_n(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \end{aligned} \quad (5.7)$$

where $\alpha^n = \frac{1}{1+n|u_n-h|}$ and C, R are CAFs associated with Φ^+, Φ^- (see (4.4)). By Theorem 3.2, $u_n(s, x) = u_n(s, X_s) \geq u_{n+1}(s, X_s) = u_{n+1}(s, x)$ and $u_n(s, x) = u_n(s, X_s) \geq h(s, X_s) = h(s, x)$, $P_{s,x}$ -a.s. for a.e. $(s, x) \in Q_{\hat{T}}$. This means that u_n is convergent almost everywhere. Set $u(s, x) = \lim_{n \rightarrow \infty} u_n(s, x)$ if the limit exists and is finite, and $u(s, x) = 0$ otherwise. Of course, $u \geq h$ a.e.. By standard calculations, taking u_n as a test function in (5.4), we get

$$\|u_n(t)\|_{2,\rho}^2 + \|\nabla u_n\|_{2,\rho,T}^2 \leq C(\|\varphi\|_{2,\rho}^2 + \|g\|_{2,\rho}^2 + \|\Phi^-\|_{\mathbb{L}_2(0,T;H_\rho^{-1})}^2). \quad (5.8)$$

By (3.7), for a.e. $(s, x) \in Q_{\hat{T}}$,

$$\begin{aligned} E_{s,x} |(u_n - u_m)(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla(u_n - u_m)|^2(\theta, X_\theta) d\theta \\ \leq E_{s,x} \int_s^T |\alpha^n - \alpha^m| |u_n - u_m|(\theta, X_\theta) dR_{s,\theta} \end{aligned} \quad (5.9)$$

for all $t \in [s, T]$. Multiplying this inequality by ρ^2 and using [12, Proposition 4.1] (see also [3]) we obtain

$$\begin{aligned} \|u_n(t) - u_m(t)\|_{2,\rho}^2 + \|\sigma \nabla u_n - \sigma \nabla u_m\|_{2,\rho,T}^2 \\ \leq C E_{s,\rho} \int_s^T |(\alpha^m - \alpha^n)| |u_n - u_m|(\theta, X_\theta) dR_{s,\theta}, \end{aligned}$$

for every $t \in [s, T]$. Here $E_{s,\varrho}$ denotes the integral with respect to the measure $P_{s,\varrho}$, where $P_{s,\varrho}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \varrho(x) dx$. Next,

$$\begin{aligned} E_{s,x} \int_s^T |\alpha^n - \alpha^m| |u_n - u_m|(\theta, X_\theta) dR_{s,\theta} \\ \leq 2E_{s,x} \int_s^T (|h| + |u_1|)(\theta, X_\theta) dR_{s,\theta} \\ \leq 2E_{s,x} \sup_{s \leq t \leq T} (|h| + |u_1|)^2(t, X_t) + 2E_{s,x} |R_{s,T}|^2. \end{aligned}$$

By Corollaries 3.4 and 3.6 in [12], $E_{s,\varrho} \sup_{s \leq t \leq T} (|h| + |u_1|)^2(t, X_t) + E_{s,\varrho} |R_{s,T}|^2 < \infty$. On the other hand, by Theorem 3.2, for a.e. $(s, x) \in Q_{\hat{T}}$ the right-hand side of (5.9) tends to zero when $n, m \rightarrow \infty$. Therefore applying the Lebesgue dominated convergence theorem we conclude that $u_n \rightarrow u$ in $\mathbb{L}_2(0, T; H_\varrho^1) \cap C([0, T], \mathbb{L}_{2,\varrho}(\mathbb{R}^d))$. Furthermore, if $\eta \in \mathbb{L}_2(0, T; H_\varrho^1)$, then by (5.4),

$$\begin{aligned} |\langle \frac{\partial u_n}{\partial t}, \eta \rangle_\varrho| \leq C \{ \|\nabla u_n\|_{2,\varrho,T} \|\nabla \eta\|_{2,\varrho,T} + (\|g\|_{2,\varrho,T} + \|u_n\|_{2,\varrho,T} + \|\nabla u_n\|_{2,\varrho,T}) \|\eta\|_{2,\varrho,T} \\ + \|\Phi^-\|_{\mathbb{L}_2(0,T;H_\varrho^{-1})} \|\eta\|_{\mathbb{L}_2(0,T;H_\varrho^1)} \}. \end{aligned}$$

By (5.8) and the assumptions on the barrier h , the right-hand side of the above inequality is bounded. Hence, by the Banach-Steinhaus theorem, $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $\mathbb{L}_2(0, T; H_\varrho^{-1})$. Consequently, there is a subsequence (still denoted by $\{n\}$) such that $\frac{\partial u_n}{\partial t} \rightarrow \frac{\partial u}{\partial t}$ weakly in $\mathbb{L}_2(0, T; H_\varrho^{-1})$. By the above and (5.4) we conclude that

$$\frac{\partial u}{\partial t} + L_t u = -f_u - \mu, \quad u(T) = \varphi, \quad (5.10)$$

where μ is a weak limit of $\{\mu_n\}$ in $\mathbb{L}_2(0, T; H_\varrho^{-1})$. On the other hand, passing to the limit in (5.7) we conclude that there is an increasing process $K^{s,x}$ on $[s, T]$ such that $E_{s,x} |K_T^{s,x}|^2 < \infty$ and

$$\begin{aligned} u(t, X_t) = \varphi(X_T) + \int_t^T f_u(\theta, X_\theta) d\theta + K_T^{s,x} - K_t^{s,x} \\ - \int_t^T \sigma \nabla u(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \end{aligned}$$

for a.e. $(s, x) \in Q_{\hat{T}}$. By the form of above equation we can drop dependence on (s, x) in notation. This proves (5.5) by Theorem 3.2. Formula (5.6) follows from Theorem 4.6. Now, let us fix $\eta \in \mathcal{W}_\varrho$. From (5.4), (5.10) we get

$$\begin{aligned} |\langle \mu_n - \mu, \eta \rangle_{\varrho,T}| \leq C \{ (\|u_n - u\|_{2,\varrho,T} + \|\nabla u_n - \nabla u\|) \|\eta\|_{2,\varrho,T} \\ + \|\nabla u_n - \nabla u\|_{2,\varrho,T} \|\nabla \eta\|_{2,\varrho,T} + \|\frac{\partial \eta}{\partial t}\|_{\mathbb{L}_2(0,T;H_\varrho^{-1})} \|u_n - u\|_{\mathbb{L}_2(0,T;H_\varrho^1)} \\ + \sup_{t \in [0,T]} \|u_n(t) - u(t)\|_{2,\varrho} \|\eta(t)\|_{2,\varrho} \}. \end{aligned}$$

Since we know that $u_n \rightarrow u$ in $\mathbb{L}_2(0, T; H_\varrho^1) \cap C([0, T], \mathbb{L}_{2,\varrho}(\mathbb{R}^d))$, $\mu_n \rightarrow \mu$ in \mathcal{W}'_ϱ . The proof is completed by showing that the pair (u, μ) is a solution of (1.8). By what

has already been proved all conditions of the definition of the solution are satisfied but the last condition in (5.3). To this end we first note that by the definition of the solution of RBSDE, $E_{s,x} \int_s^T (u - h)(\theta, X_\theta) dK_{s,\theta} = 0$ for a.e. $(s, x) \in Q_{\hat{T}}$. Next, let us observe that formula (5.6) can be extended to functions ξ such that one of the integrals appearing in (5.6) is finite. In particular, by [12, Corollary 3.6], (5.6) holds for $\xi = u - h$. Consequently, for a.e. $s \in [0, T]$ we have

$$0 = \int_{\mathbb{R}^d} \left(E_{s,x} \int_s^T (u - h)(\theta, X_\theta) dK_{s,\theta} \right) dx \geq C \int_s^T \int_{\mathbb{R}^d} (u - h) d\mu,$$

the last inequality being a consequence of Aronson's lower estimate (see [2]). This and the fact that any measure in $\mathbb{L}_2(0, T; H_\varrho^{-1})$ vanishes on the sets of the form $\{t\} \times \mathbb{R}^d$ (see comments at the beginning of section 4) completes the proof. \square

In [20] existence of strong solutions of variational problems more general than (1.7) (with nonlinear operator L_t) is proved by using a different sort of homographic approximation. Let us point out that contrary to [20] our homographic approximation of u is direct in the sense that to define the approximating sequence $\{u_n\}$ we need not to approximate the functional $(\frac{\partial h}{\partial t} + L_t h + f_h)^-$ by elements of $\mathbb{L}_{2,\varrho}(Q_T)$. Secondly, we have proved strong convergence in $\mathbb{L}_{2,\varrho}(Q_T)$ of gradients of the approximating sequence to the gradient of u .

Corollary 5.3. *Assume that (H1), (H2), (H5), (H6) are satisfied. If $\frac{\partial h}{\partial t} + L_t h \in \mathcal{M}(Q_T)$, $(\frac{\partial h}{\partial t} + L_t h - f_h)^- \in \mathbb{L}_{2,\varrho}(Q_T)$ and (u, μ) is a solution of (1.8), then $d\mu = r dm$, where*

$$r(t, x) = \alpha(t, x) \left(\frac{\partial h}{\partial t} + L_t h + f_h \right)^-(t, x) \quad (5.11)$$

for some measurable function α such that

$$\alpha(t, x) \mathbf{1}_{\{u=h\}}(t, x) = \alpha(t, x), \quad 0 \leq \alpha(t, x) \leq 1$$

for a.e. $(t, x) \in Q_T$.

Proof. This is an immediate consequence of Theorem 5.2 since α_n is bounded in $\mathbb{L}_{2,\varrho}(Q_T)$ and $(\frac{\partial h}{\partial t} + L_t h + f_h)^- \in \mathbb{L}_{2,\varrho}(Q_T)$. The fact that $\alpha(t, x) \mathbf{1}_{\{u=h\}}(t, x) = \alpha(t, x)$ follows from uniqueness of the solution of the obstacle problem because the measure μ is supported in $\{u = h\}$. \square

Corollary 5.4. *Under the assumptions of Corollary 5.3 there exists a version u of the first component of the solution of (1.8) such that the triple*

$$\left(u(t, X_t), \sigma \nabla u(t, X_t), \int_s^t r(\theta, X_\theta) d\theta \right), \quad t \in [s, T],$$

where r is defined by (5.11), is a solution of RBSDE (1.9) for a.e. $(s, x) \in Q_{\hat{T}}$.

Remark 5.5. Let us assume that the operator L_t is symmetric and let assumptions of Theorem 5.2 hold. By Corollary 3.3, for every $\eta \in C_0^\infty(Q_T)$ we have

$$0 \leq E_{s,x} \int_s^T \eta(t, X_t) dK_{s,t} \leq E_{s,x} \int_s^T \mathbf{1}_{\{u(t, X_t)=h(t, X_t)\}} \eta(t, X_t) dR_{s,t}$$

for a.e. $(s, x) \in Q_T$. Integrating the above inequality over Q_T with respect to (s, x) and using (4.7), (5.6) and the symmetry of L_t we obtain

$$0 \leq \left(\frac{\partial u}{\partial t} + L_t u + f_u \right) \leq \mathbf{1}_{\{u=h\}} \left(\frac{\partial h}{\partial t} + L_t h + f_h \right)^-,$$

i.e. the Lewy-Stampacchia inequality for solutions of the problem (5.2), (5.3) (see [5, 20] and [16] for the inequality for solutions of elliptic equations).

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